The Relation between Wind Velocity at 1000 *Metres Altitude and the Surface Pressure Distribution.*

By E. Gold, M.A., Fellow of St. John's College, Cambridge.

(Communicated by Dr. W. N. Shaw, F.R.S. Received February 25,—Read March 5, 1908.)

For the steady horizontal motion of air along a path whose radius of curvature is r, we may write directly the equation

$$\frac{(\omega r \sin\lambda \pm v)^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{(\omega r \sin\lambda)^2}{r},$$

expressing the fact that the part of the centrifugal force arising from the motion of the wind is balanced by the effective gradient of pressure.

In the equation p is atmospheric pressure, ρ density, v velocity of moving air, λ is latitude, and ω is the angular velocity of the earth about its axis.

If $\partial p/\partial r$ be negative, it is clear that v and $\omega r \sin\lambda$ must have opposite signs: or, for motion in a path concave towards the higher pressure, the air must rotate in a clockwise direction, the well-known result for anticyclonic motion. Further, the maximum numerical value of $\frac{1}{\rho}\frac{\partial p}{\partial r}$ is $\frac{(\omega r \sin\lambda)^2}{r}$ and the corresponding maximum value for v is $\omega r \sin\lambda$. Therefore, in anticyclonic regions there are limiting values which the gradient and the velocity cannot exceed. This limiting value of v for latitude 50° and $r = 100$ miles is approximately 20 miles per hour.

At the surface of the earth, owing to friction and eddies, the mean direction of the motion of the air is nearly always inclined to the isobars; but over the sea the inclination is very much less, and it seemed probable that in the upper regions of the atmosphere, if the motion were steady, the air would in general move tangentially to the isobars, and its velocity would agree with that calculated from the equation given above.

The question, however, arises as to whether the pressure is likely to continue steady long enough for a condition in which the equation is applicable to supervene. We can get an idea of the time that would elapse before air, starting from rest, would reach a state of steady motion, by considering the motion of a particle on the earth's surface (1) under a constant force in a constant direction, corresponding to straight isobars; (2) under a constant radial force corresponding to cyclonic and anticyclonic conditions. The particle would begin to move at right angles to the isobars in the

direction of the force, but as its velocity increased it would be deflected by the effect of the earth's rotation until it moved perpendicularly to the force.

The equations of motion of a particle, referred to axes fixed relatively to the earth and having an origin on the surface in latitude λ, are

$$\frac{d^2x}{dt^2} - 2\omega\cos\lambda\frac{dz}{dt} - 2\omega\sin\lambda\frac{dy}{dt} = X,$$

$$\frac{d^2y}{dt^2} + 2\omega\sin\lambda\frac{dx}{dt} = Y,$$

$$\frac{d^2z}{dt^2} + 2\omega\cos\lambda\frac{dx}{dt} = Z,$$

where the axis of z is vertical and the axes of x and y are west and south respectively.

If there is no vertical motion we may write the first two equations

$$\frac{d^2x}{dt^2} - a\frac{dy}{dt} = X, \qquad \frac{d^2y}{dt^2} + a\frac{dx}{dt} = Y,$$

and the form of the equations and the value of a are unaltered by changing to other axes in the same plane. Let us take the y axis to be in the direction of the constant force b. Then

$$\frac{d^2x}{dt^2} - a\frac{dy}{dt} = 0, \qquad \frac{d^2y}{dt^2} + a\frac{dx}{dt} = b,$$

whence
$$x = \frac{b}{a^2}(at - \sin at), \qquad y = \frac{b}{a^2}(1 - \cos at),$$

if the particle start from rest. The motion is therefore oscillatory, and the particle moves in a series of cycloidal-like curves, fig. 1. The times to the successive intersections with $y = b/a^2$ are $\pi/2a$, $3\pi/2a$, etc. For latitude 50°

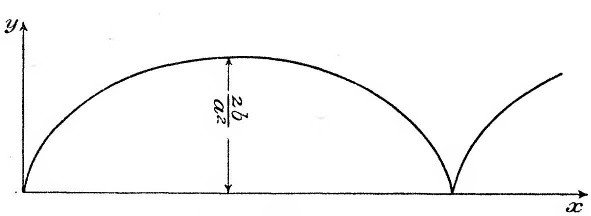


Fig. 1.

these are about 4 and 12 hours. They are independent of b. If there is damping, the motion will be as in fig. 2. If the motion is resisted by a force kv proportional to the velocity, the path will be inclined to the x-axis. Fig. 3 gives the path for the particular case $k=a$ and for a period of time equal to $2\pi/a$, or 16 hours.

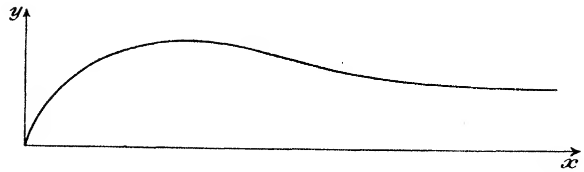


FIG. 2.

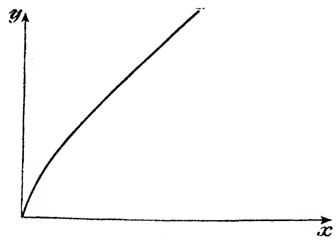


FIG. 3.

In the case of a constant radial force we have for the motion

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \mathrm{R} + ar\frac{d\theta}{dt},$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} + a\frac{dr}{dt} = 0,$$

whence $$r^2\frac{d\theta}{dt} + \tfrac{1}{2}ar^2 = \mathrm{B}.$$

If the particle start from the centre,

$$\mathrm{B} = 0 \quad \text{and} \quad \frac{d\theta}{dt} = -\tfrac{1}{2}a,$$

and we obtain $$r = \frac{4\mathrm{R}}{a^2}(1-\cos\tfrac{1}{2}at) = \frac{4\mathrm{R}}{a^2}(1-\cos\theta).$$

The particle therefore describes a cardioid, but if there is damping the motion will come to be along the circle $r = 4\mathrm{R}/a^2$.

The time to reach the circle is π/a, or about 8 hours for latitude 50°.

These times are not large meteorologically, and we may therefore expect the relation between air velocity and pressure gradient to be that corresponding to steady motion so long as there are no irregularities to produce turbulent motion.

For application to wind velocities in the upper air we require to know the upper-air isobars. If we have air in which the horizontal layers are isothermal, then from the equations

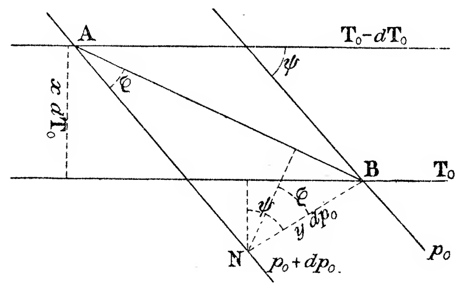
$$dp = -g\rho\, dz, \qquad p = gk\rho\mathrm{T},$$

it follows that $$\log\frac{p_0}{p_z} = \int_0^z \frac{dz}{k\mathrm{T}}.$$

We have, therefore, if p_0 and p_0+dp_0 are surface isobars and p_z and p_z+dp_z the corresponding upper isobars,

$$\frac{dp_z}{p_z}=\frac{dp_0}{p_0}, \quad \text{so that} \quad \frac{dp_z}{\rho_z}=\frac{dp_0}{\rho_0}\frac{\mathrm{T}_z}{\mathrm{T}_0}.$$

Therefore the velocity calculated from the surface isobars will apply to the upper air, except for the factor $\mathrm{T}_z/\mathrm{T}_0$. For $z = 1000$ metres the effect of this factor is to diminish the velocity by about 2 per cent.



If the conditions are not isothermal, but such that the isotherms and isobars intersect at an angle ψ, the upper isobars will have a different direction from the surface isobars, and the value of the upper gradient will also be changed.

The pressure at a height z above B, the point of intersection of p_0, T_0, is $p_0e^{-z/k\mathrm{T}_m}$, and above A, the point of intersection of p_0+dp_0, $\mathrm{T}_0-d\mathrm{T}_0$, is

$$(p_0+dp_0)\, e^{-z}/k\,(\mathrm{T}_m-d\mathrm{T}_m).$$

If we assume the vertical temperature gradient to be the same over all the region considered, $d\mathrm{T}$ will be the same for every element of the above integral, and we can put $d\mathrm{T}_m = d\mathrm{T}_0$.

If these two pressures at height z are equal, we must have

$$p_0e^{-z}/k\mathrm{T}_m = (p_0+dp_0)\, e^{-z}/k\,(\mathrm{T}_m-d\mathrm{T}_0),$$

or

$$\frac{dp_0}{p_0}=\frac{z}{k}\frac{d\mathrm{T}_0}{\mathrm{T}_m^2}, \quad \text{or} \quad \frac{dp_0}{\rho_0}=gz\,\frac{\mathrm{T}_0 d\mathrm{T}_0}{\mathrm{T}_m^2}.$$

In this case AB is the direction of the upper isobar and its inclination ϕ to the lower isobar is given by

$$\tan\phi = \frac{y dp_0}{x\, d\mathrm{T}_0.\mathrm{cosec}\,\psi + y\, dp_0 \cot\psi},$$

where $x\, d\mathrm{T}_0$ and $y\, dp_0$ are the distances between the isotherms and isobars.

Substituting for $d\mathrm{T}_0$ and dividing out by dp_0, we get

$$\cot\phi = \cot\psi + \frac{x}{y}\frac{\mathrm{T}_m^2}{g\rho_0 z\mathrm{T}_0}\,\mathrm{cosec}\,\psi.$$

Taking y and x for millimetre isobars and $1°$ C. isotherms and putting $z = 1000$ metres and $T_m^2/T_0 = 2T_m - T_0 = 270°$ C., say, we find

$$\cot\phi = \cot\psi + 2{\cdot}8\frac{x}{y}\operatorname{cosec}\psi.$$

To obtain the upper pressure gradient, we consider the upper isobars over B and N. The difference of temperature between B and N is $y/x\,.\,dp_0\cos\psi = dt$, say.

Therefore the upper pressure difference is

$$(p_0+dp_0)\,e^{-z}/k\,(T_m+dt) - p_0e^{-z}/kT_m = e^{-z}/kT_m\left[dp_0 + \frac{p_0zdt}{kT_m^2}\right]$$

$$= p_0{}^{-z}/kT_m\left[1 + \frac{p_0z\,.\,y\cos\psi}{xkT_m^2}\right].$$

The distance between these isobars is $ydp_0\cos\phi$ and the upper gradient is consequently

$$\frac{1}{y\cos\phi}e^{-z}/kT_m\left[1+\frac{p_0zy\cos\psi}{xkT_m^2}\right] = \frac{1}{y\cos\phi}e^{-z}/kT_m\left[1+\frac{g\rho_0zT_0}{T_m^2}\frac{y\cos\psi}{x}\right],$$

and the ratio $\dfrac{1}{\rho_z}\dfrac{\partial p_z}{\partial r}\Big/\dfrac{1}{\rho_0}\dfrac{\partial p_0}{\partial r}$ is $\sec\phi\left[1+\dfrac{g\rho_0zT_0}{T_m^2}\dfrac{y\cos\psi}{x}\right]\dfrac{T_z}{T_0}$,

which is $g\rho z\operatorname{cosec}\phi\,.\,\dfrac{y\sin\psi}{x}$, taking T_z/T_0 to be unity, namely,

$$\operatorname{cosec}\phi\,.\,\frac{1}{\cot\phi-\cot\psi}\quad\text{or}\quad\frac{\sin\psi}{\sin(\psi-\phi)}.$$

In the special cases, $\psi = 0$ or $180°$, the ratios are

$$\left(1\pm\frac{g\rho_0z}{T}\,.\,\frac{y}{x}\right)\quad\text{or}\quad\left(1\pm\frac{y}{2{\cdot}8x}\right),\ \text{for } z = 1000 \text{ metres.}$$

If $x = 2y$, which would represent a possible case, the increase or decrease would be about 18 per cent.

For $\phi = \frac{1}{2}\pi$ the rotation would in the same circumstances be about $10°$.

During the year 1905 a series of observations in the upper air was made at Berlin and Lindenberg, near the time of the general 8 A.M. morning observations. It was therefore possible to compare the wind velocities observed with those calculated from measurements of the gradient by the use of the formula at the beginning of this paper, the motion being assumed tangential to the isobars.

For purposes of calculation the formula may be written

$$v\,(1\pm0{\cdot}00108\,v\cot\psi\operatorname{cosec}\lambda) = \frac{709\operatorname{cosec}\lambda}{x}\frac{T}{T_0}\frac{B_0}{B},$$

where ψ is the angular radius of the small circle, on the earth's surface,

osculating the path, v is in metres per second, x is the distance in kilometres between millimetre isobars, T, B are the temperature and pressure, and T_0, B_0 the corresponding values for air at 0° C. and 760 mm.

If the motion is along straight lines, $\cot\psi = 0$, and the values of v for $B = B_0$, $T = T_0$, are as follows if $x = 50$ kilometres.

Latitude	30°	40°	50°	60°	70°
v	28·4	22·1	18·5	16·4	15·1

If v_0 represent the velocity when $\cot\psi = 0$, we can most easily express the solutions of the equation for different values of ψ, x, λ, by taking as independent variables, ψ, v_0, λ.

Taking, as an example of the dependence on ψ, $\lambda = 50°$, $v_0 = 40$ metres per second, we obtain the following values for v in metres per second in the case of cyclonic motion.

ψ.........	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
v	17	21	24	26	28	29	30	31	31	32

For anticyclonic motion the gradient corresponding to $v_0 = 40$ metres per second is above the maximum, and we take for two examples $v_0 = 12$, 30 metres per second.

The values of v are then as follows for the two cases :—

	ψ.	1°.	2°.	3°.	4°.	5°.	6°.	7°.	8°.	9°.	10°.	
For $v_0 = 12$.........	v =	—	—	—	20	16	15	14	14	14	13	m. p. s.
For $v_0 = 30$.........	v =	—	—	—	—	—	—	—	—	—	50	,,

Where no value is inserted for v, the gradient corresponding to the given value of v_0 is above the maximum for the corresponding value of ψ.

To show the dependence on λ, we take $\psi = 3°$, and put $v_0 = 40$ metres per second for cyclonic motion, and $v_0 = 10$, 5 metres per second for anticyclonic motion. The following table gives the values of v for different latitudes in the three cases :—

	λ.	30°.	40°.	50°.	60°.	70°.	
For $v_0 = 40$	v =	21	23	24	25	26	m. p. s.
For $v_0 = 10$	v =	—	—	—	17	15	,,
For $v_0 = 5$	v =	7·1	6·3	6·0	5·8	5·7	,,

By the use of tables giving values of v_0 for different values of x, T, B, and of v for different values of λ, v_0, ψ, each wind observation at 1000 metres altitude was compared with the value deduced from the surface isobars. The temperature correction was not applied.

The following table gives the result of the comparisons :—

Comparison between Barometric Gradient Wind and Observed Wind at 1000 Metres Altitude for the Year 1905 (Berlin and Lindenberg).

Table of Averages.

	Theoretical velocity v. m.p.s.	Velocity v_1 at 1000 metres. m.p.s.	Percentage difference between v and v_1.	Mean deviation in points of v_1 from v (+ when v_1 veers from v).	Mean surface wind. m.p.s.	Mean deviation in points of wind v_1 from surface wind.	Surface wind ÷ v cosine (deviation).	Wind at 1000 metres ÷ v_1 cosine (deviation).	Component wind ⊥ to isobars at surface ÷ component at 1000 metres.
Berlin.									
January	15·7	15·1	4·0	−0·6	5·0	3·3	0·45	0·97	1·9
February	12·0	11·6	3·5	−0·6	4·0	2·7	0·41	0·98	1·7
March	8·1	6·6	23·0	−0·4	2·6	1·7	0·35	0·82	1·9
Lindenberg.									
April	10·2	8·8	16·0	−1·0	4·8	1·7	0·54	0·90	1·4
May	8·3	6·5	28·0	−1·0	4·5	2·3	0·65	0·80	1·9
June	6·8	6·4	6·0	−0·5	3·9	1·1	0·60	0·95	1·9
July	8·4	8·2	2·4	−0·9	4·5	1·2	0·58	1·0	1·3
August	8·9	8·0	11·0	−0·9	5·1	1·6	0·64	0·92	1·6
September	10·3	10·5	−2·0	−0·8	5·4	2·2	6·63	1·03	1·6
October	12·1	11·9	2·0	−0·5	6·3	2·7	0·64	0·99	2·7
November	10·6	10·1	5·0	−0·9	5·9	3·1	0·78	0·97	2·3
December	11·0	10·3	7·0	+0·3	5·5	3·9	0·65	0·94	−5·−?
Summer	8·8	8·1	8·5	−0·8	4·7	1·7	—	0·93	—
Winter	11·6	10·9	6·5	−0·45	4·9	2·9	—	0·95	—
Year	10·2	9·5	7·5	−0·6	4·8	2·3	—	0·94	—

The upper wind coincides in direction very nearly with the isobars at the surface, and the wind velocity observed agrees well with that calculated from the pressure distribution. The differences are not greater than possible errors of observation, except in spring.

It is known that the upper wind always veers from the surface wind, and the numbers in Column 7 show that in 1905 the veering was considerably greater in winter than in summer.

If the effect of the earth's surface were the same as if a frictional force opposed the motion, the relation between the wind and gradient of pressure would be as above, except that the effective gradient would be the maximum gradient multiplied by the cosine of α, the angle between the path and the isobars. The corresponding velocity would be approximately $v \cos \alpha$, except in cases of considerable curvature. In the majority of the observations the curvature was small, and we should therefore expect the surface wind to be nearly $v \cos \alpha$, so that the numbers in Column 8 would be nearly unity. This is far from being the case; but the change of the station of observation from Berlin to Lindenberg is accompanied by a corresponding change in the ratio of the surface wind velocity to $v \cos \alpha$.

This suggests that the effect of the surface, apart from the purely frictional effect, is to reduce the velocity in a given direction in a constant ratio depending on the locality, and that departures in the observed velocities from those corresponding to this ratio are to be associated with unsteady meteorological conditions.

The last column gives approximately the ratio of the volume of air crossing the isobars at the surface to the volume crossing at 1000 metres.

The ratio appears to be nearly constant; the change in December is probably due to the exceptional conditions which prevailed during part of the month, when the air was considerably warmer at 1000 metres altitude than at the surface.